Transforming a scale into another

For linear scales we can use the following formula to transform X into Y:

$$Y(X)=a_Y\cdot X+b_Y$$

We have to determine the two unknown constants a and b and therefore give two equations defining two values in our new system:

$$Y_1=a_Y\cdot X_1+b_Y$$
$$Y_2=a_Y\cdot X_2+b_Y$$

This yields

$$a_Y=\frac{Y_2-Y_1}{X_2-X_1}=\frac{\Delta Y}{\Delta X}$$
$$b_Y=Y_2-\frac{Y_2-Y_1}{X_2-X_1}\cdot X_2=Y_2-\frac{\Delta Y}{\Delta X}\cdot X_2$$

The whole formula thus can be written as

$$Y(X)=\frac{Y_2-Y_1}{X_2-X_1}\cdot X+Y_2-\frac{Y_2-Y_1}{X_2-X_1}\cdot X_2$$

The unit itself (e.g. „°C“) is here described with [X], [Y] for the respective system.
For our constants we have

$[a_Y]=\frac{[Y]}{[X]}$ and $[b_Y]=[Y]$

Transforming back

If we want express our new unit Y by X we get

$$X(Y)=a_Y{}'\cdot Y+b_Y{}'$$

Using our known furmals from above this equals

$$X(Y)=\frac{Y-b_Y}{a_Y}=\frac{Y}{a_Y}-\frac{b_Y}{a_Y}=\frac{X_2-X_1}{Y_2-Y_1}\cdot Y-X_2+\frac{X_2-X_1}{Y_2-Y_1}\cdot Y_2$$

This gives us the relations

$$a_Y{}'=\frac{1}{a_Y}=\frac{X_2-X_1}{Y_2-Y_1}=\frac{\Delta X}{\Delta Y}$$
$$b_Y{}'=-\frac{b_Y}{a_Y}=-X_2+\frac{X_2-X_1}{Y_2-Y_1}\cdot Y_2=-X_2+\frac{\Delta X}{\Delta Y}\cdot Y_2$$

$[a_Y{}']=\frac{[X]}{[Y]}$ and $[b_Y{}']=[X]$

Transforming a transformed unit

We want to express unit Z

$$Z(X)=a_Z\cdot X+b_Z$$

as a function of Y:

$$Z(Y)=\tilde{a}_Z\cdot Y+\tilde{b}_Z=\frac{Z_2-Z_1}{Y_2-Y_1}\cdot Y+Z_2-\frac{Z_2-Z_1}{Y_2-Y_1}\cdot Y_2$$

Using our knowledge from the first paragraph we can write

$$Z(Y)=\tilde{a}_Z\cdot(a_Y\cdot X+b_Y)+\tilde{b}_Z=a_Y\cdot\tilde{a}_Z\cdot X+b_Y\cdot\tilde{a}_Z+\tilde{b}_Z$$

which leads to

$a_Z=a_Y\cdot\tilde{a}_Z$ and $b_Z=b_Y\cdot\tilde{a}_Z+\tilde{b}_Z$

$[a_Z]=\frac{[Z]}{[X]}$ and $[b_Z]=[Z]$

$[\tilde{a}_Z]=\frac{[Z]}{[Y]}$ and $[\tilde{b}_Z]=[Z]$

Example

world generator uses temperatures X, [X]="" (no unit name, only values 0...1)

we want to transform into Y, [Y]="°C"

We have to give two equations:

(A) 0°C = 0.3

(B) 30°C = 0.7

This means Y_1 = 0°C, Y_2 = 30°C, X_1 = 0.3, X_2 = 0.7

$$Y(X)=a_Y \cdot X+b_Y=\frac{Y_2-Y_1}{X_2-X_1}\cdot X+Y_2-\frac{Y_2-Y_1}{X_2-X_1}\cdot X_2=(75\cdot X-22.5)\,^\circ C$$

$$a_Y=75\,\frac{^\circ C}{1} \text{ and } b_Y=-22.5\,^\circ C$$

Now we want to use the Fahrenheit temperatur scale with

(A) 32°F = 0°C

(B) 212°F = 100°C

$$Z(Y)=\tilde{a}_Z\cdot Y+\tilde{b}_Z=\frac{Z_2-Z_1}{Y_2-Y_1}\cdot Y+Z_2-\frac{Z_2-Z_1}{Y_2-Y_1}\cdot Y_2=(1.8\cdot Y+32)\,^\circ F$$

$$\tilde{a}_Z=1.8\,\frac{^\circ F}{^\circ C} \text{ and } \tilde{b}_Z=32\,^\circ F$$

To calculate Fahrenheit by world generator temperature:

$$a_Z=a_Y\cdot\tilde{a}_Z=135\,\frac{^\circ F}{1} \text{ and } b_Z=b_Y\cdot\tilde{a}_Z+\tilde{b}_Z=-8.5\,^\circ F$$

getting the function

$$Z(X)=(135\cdot X-8.5)\,^\circ F$$

Let's say we want to know what temperature we would have to use in the world generator to get a temperature of 60 °F (here we have to use Z for Fahrenheit instead of Y because of our definition):

$$X(Z)=\frac{Z-b_Z}{a_Z}=\frac{Z}{a_Z}-\frac{b_Z}{a_Z}=\frac{1}{135\,^\circ F}\cdot Z+\frac{17}{270}$$

$$X(60\,^\circ F)\approx 0.507$$